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Asymptotically linear Schrödinger equation with potential vanishing at infinity

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Abstract

We are concerned with the existence of bound states and ground states of the following nonlinear Schrödinger equation

$$\begin{cases} -\Delta u(x) + V(x)u(x) = K(x)f(u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad u(x) > 0, & N \geqslant 3, \end{cases} \tag{0.1}$$

where the potential $V(x)$ may vanish at infinity, $f(s)$ is asymptotically linear at infinity, that is, $f(s) \sim O(s)$ as $s \to +\infty$. For this kind of potential, it seems difficult to find solutions in $H^1(\mathbb{R}^N)$, i.e. bound states of (0.1). If $f(s) = s^p$ and $p \in (\sigma, (N+2)/(N-2))$ with $\sigma \geqslant 1$, Ambrosetti, Felli and Malchiodi [A. Ambrosetti, V. Felli, A. Malchiodi, Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity, J. Eur. Math. Soc. 7 (2005) 117–144] showed that (0.1) has a solution in $H^1(\mathbb{R}^N)$ and (0.1) has no ground states if p is out of the above range. In this paper, we are interested in what happens if $f(s)$ is asymptotically linear. Under appropriate assumptions on K, we prove that (0.1) has a bound state and a ground state.

MSC: 35J20; 35J60

Keywords: Nonlinear Schrödinger equation; Vanishing potential; Bound state

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1. Introduction

In this paper, we consider the following stationary nonlinear Schrödinger equation

$$-\Delta u(x)+V(x)u(x)=K(x)f(u), \quad x\in\mathbb{R}^N,\ N\geqslant 3, \tag{1.1}$$

where the functions V, f and K satisfy the following conditions:

(V$_1$) $V:\mathbb{R}^N\to\mathbb{R}$ is continuous and there exist $a, A, \alpha>0$ such that

$$\frac{a}{1+|x|^\alpha}\leqslant V(x)\leqslant A.$$

(F$_1$) $f\in C(\mathbb{R},\mathbb{R}^+)$, $f(s)\equiv 0$ for all $s\leqslant 0$ and $f(s)s^{-1}\to 0$ as $s\to 0^+$.
(F$_2$) There exists $l\in(0,+\infty)$ such that $f(s)s^{-1}\to l$ as $s\to\infty$.
(K$_1$) K is a positive continuous function and there exists $R_0>0$ such that

$$\sup\left\{\frac{f(s)}{s}\colon s>0\right\}<\inf\left\{\frac{V(x)}{K(x)}\colon |x|\geqslant R_0\right\}.$$

It is easy to see that the condition (K$_1$) can be obtained by assuming that

$$\sup\left\{\frac{f(s)}{s}\colon s>0\right\}<\liminf_{|x|\to+\infty}\left\{V(x)/K(x)\right\}. \tag{1.2}$$

Remark 1.1. In paper [1], the condition (V$_1$) is also assumed, but the condition on K is as follows

(K$_2$) $K:\mathbb{R}^N\to\mathbb{R}$ is smooth and there exist $k,\beta>0$ such that

$$0<K(x)\leqslant\frac{k}{1+|x|^\beta}. \tag{1.3}$$

Clearly, if (K$_2$) holds with $0<\alpha<\beta$, then $\lim_{|x|\to+\infty}V(x)/K(x)=+\infty$. By (1.2), we see that the condition (K$_2$) with $\alpha\in(0,\beta)$ leads to the condition (K$_1$).

Here are two examples, in which our conditions (V$_1$), (F$_1$), (F$_2$) and (K$_1$) are satisfied, but (K$_2$) fails.

Example 1.1. Let $V(x)=\frac{1}{\ln\ln(3+|x|)}$ and $K(x)=\frac{1}{\ln(3+|x|^2)}$, we see that (V$_1$) holds and $\lim_{|x|\to+\infty}\frac{V(x)}{K(x)}=+\infty$, then it is easy to see there is f such that (F$_1$), (F$_2$) and (K$_1$) hold. But in this case, K does not satisfy (K$_2$).

Example 1.2. For any fixed $R_0>0$, let $V(x)=1/\ln(3+|x|)$ and

$$K(x)=\begin{cases}V(x)/(1+|x|), & \text{if } |x|<R_0,\\ V(x)/(1+R_0), & \text{if } |x|\geqslant R_0,\end{cases}\qquad f(s)=\begin{cases}R_0s^2/(1+s), & \text{if } s>0,\\ 0, & \text{if } s\leqslant 0.\end{cases}$$

Since $V(0) = 1/\ln 3$ and $\lim_{|x|\to+\infty} \frac{1+|x|^\alpha}{\ln(3+|x|)} = +\infty$ for any $\alpha > 0$, it is easy to see that there exists $a > 0$ such that (V_1) holds, but (K_2) is false. Note that

$$\sup\left\{\frac{f(s)}{s}\colon s > 0\right\} = R_0 < R_0 + 1 = \inf\left\{\frac{V(x)}{K(x)}\colon |x| \geqslant R_0\right\},$$

then (K_1) holds. Moreover, by the definition of f, (F_1) and (F_2) are satisfied.

Throughout this paper, we define the following weighted Sobolev space

$$H = \left\{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\colon \int_{\mathbb{R}^N} \left[|\nabla u|^2 + V(x)u^2\right] dx < \infty\right\}.$$

Clearly, $H^1(\mathbb{R}^N) \subset H$. H is a Hilbert space, its scalar product and norm are given by

$$(u, v) = \int_{\mathbb{R}^N} \left[\nabla u \nabla v + V(x)uv\right] dx \quad \text{and} \quad \|u\|_H^2 = \int_{\mathbb{R}^N} \left[|\nabla u|^2 + V(x)u^2\right] dx.$$

Furthermore, we define the energy functional $I : H \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla u|^2 + V(x)u^2\right] dx - \int_{\mathbb{R}^N} K(x)F(u)\, dx, \quad F(u) = \int_0^u f(t)\, dt. \tag{1.4}$$

By (V_1) and (K_1), there exists $C_1 > 0$ such that

$$K(x) \leqslant C_1 V(x), \quad \text{for all } x \in \mathbb{R}^N. \tag{1.5}$$

So, I is well defined on H and $I \in C^1(H, \mathbb{R})$ with

$$\left\langle I'(u), v\right\rangle = \int_{\mathbb{R}^N} \left[\nabla u \nabla v + V(x)uv\right] dx - \int_{\mathbb{R}^N} K(x)f(u)v\, dx, \quad \text{for all } v \in H.$$

Definition 1.1. A function $u \in H$ is said to be a solution of problem (1.1) provided $u \not\equiv 0$ and satisfies

$$\int_{\mathbb{R}^N} \left[\nabla u \nabla \phi + V(x)u\phi\right] dx = \int_{\mathbb{R}^N} K(x)f(u)\phi\, dx, \quad \text{for all } \phi \in H. \tag{1.6}$$

If a solution u of (1.1) is in $L^2(\mathbb{R}^N)$, we call it as a bounded state of (1.1). Moreover, a function u is a ground state of problem (1.1) we mean that u is such a solution of (1.1) which has the least energy among all solutions of (1.1), that is, $I'(u) = 0$ and $I(u) = \inf\{I(v)\colon v \in H \setminus \{0\}$ and $I'(v) = 0\}$.

In the past two decades, much attention has been paid to the existence of the bound states for superlinear problem (1.1) under various assumptions on $V(x)$. For example, if $V(x)$ satisfies

(V_2) there exists $V_0 > 0$ such that $V(x) \geqslant V_0$ for all $x \in \mathbb{R}^N$, and
(V_3) $\lim_{|x|\to+\infty} V(x) = V(\infty) \in (0, +\infty)$,

then, by the well-known concentration compactness principle [17], it is shown that there is a bound state for problem (1.1), see [2,13,14,16,19,20] and the references therein. If $V(x)$ satisfies (V_2) and $\lim_{|x|\to+\infty} V(x) = +\infty$, Rabinowitz [22] proved that (1.1) has also a bound state by a variant version of mountain pass theorem.

Recently, under the assumptions (V_1) with $0 < \alpha < 2$ and (K_2) given by (1.3), Ambrosetti, Felli and Malchiodi in [1] proved that problem (1.1) has a bound state for $f(u) = u^p$ with $\sigma < p < \frac{N+2}{N-2}$ and

$$\sigma = \begin{cases} \frac{N+2}{N-2} - \frac{4\beta}{\alpha(N-2)} > 1, & 0 < \beta < \alpha, \\ 1, & \beta \geqslant \alpha. \end{cases} \tag{1.7}$$

Moreover, it is also proved in [1] that, if $f(u) = u^p$ in (1.1), then the restriction of $p \in (\sigma, \frac{N+2}{N-2})$ is necessary to get a *ground state* (i.e. a least energy solution) of (1.1). A nature question is to ask what would happen if $f(s)/s$ tends to a constant as $s \to +\infty$, that is, f is asymptotically linear in s at infinity. In this paper, we show that (1.1) still has a ground state if f is asymptotically linear at infinity. This phenomenon is quite different from the superlinear case or the asymptotically linear with constant potential case.

Since the work of Ambrosetti, Felli and Malchiodi [1], there are many papers on problem (1.1) with potential $V(x)$ vanishing at infinity, see, for example, [3–7,26]. However, in those papers, $f(s)$ is always supposed to be the form of s^p with $p \in (1, \frac{N+2}{N-2})$ and $-\Delta$ is replaced by $-\varepsilon^2\Delta$, and the authors were more interested in the semi-classical states to (1.1) for $\varepsilon > 0$ small, less interested in the existence of solutions, because under that situation it is trivial to find a mountain pass type solution in a suitable weighted Sobolev space with the help of the compactness of Sobolev embedding. However, until the paper [1] appeared it is not clear whether or not the mountain pass solution found in the weighted Sobolev space is in L^2, that is, whether the solution is a bound state. Furthermore, we should mention that it is very easy to check that a mountain pass type solution of (1.1) is indeed a ground state if $f(s)$ is superlinear. But this seems very difficult to see in the asymptotically linear case.

If $V(x) \equiv$ constant or $V(x)$ has property (V_2), problem (1.1) with asymptotically linear nonlinearities has been studied widely in recent years, see, for example, [8,9,11–13,16,18,20,23–25, 27,28], etc. To our best knowledge, it seems that there are few results on problem (1.1) in the case where f is asymptotically linear and $V(x)$ may decay to zero at infinity. So, the main aim of this paper is to find a bound state and a ground state to (1.1) in this case. Motivated by [1], we seek first a solution of (1.1) in the weighted Sobolev space H by the mountain pass theorem, then try to prove this solution is a bound state, i.e., it is in L^2. However, as mentioned above, if f is asymptotically linear, it is difficult to know whether a mountain pass solution is a ground state. So, in order to get a ground state of (1.1) we use a constrained minimization technique instead of the mountain pass theorem, see the proof of Theorem 1.2.

As is known, to seek a weak solution of (1.1) is equivalent to finding a nonzero critical point of I in H. If $f(u) = u^p$ with $p > 1$, it is easy to use the mountain pass theorem to get a (PS) sequence of I, which is bounded in H. Furthermore, if the conditions (V_1) and (K_2) are satisfied, [1, Theorem 5] shows that the embedding $H \hookrightarrow L_K^{p+1}$ is compact for $\sigma < p < \frac{N+2}{N-2}$, where σ

is given by (1.7) and L_K^{p+1} denotes the weighted space of measurable functions $u:\mathbb{R}^N\to\mathbb{R}$ satisfying

$$|u|_{p+1,K}=\left[\int_{\mathbb{R}^N}K(x)|u|^{p+1}\,dx\right]^{\frac{1}{p+1}}<\infty.$$

This ensures that the (PS) sequence converges strongly to a nontrivial solution of (1.1). However, unlike the superlinear case, to get a bounded (PS) sequence in our situation, we have to overcome simultaneously the difficulties of verifying that I satisfies the mountain pass properties and showing that the (PS) sequence is bounded in H. On the other hand, under our conditions (V_1) and (K_1), the condition (K_2) is not always satisfied, see Examples 1.1 and 1.2. So, in this paper we cannot use the above compact embedding theorem as in [1]. Motivated by [21], we establish a compactness result, see Lemma 2.7, which ensures that the (PS) sequence converges to a nontrivial solution $u\in H$. As we know, this solution is not obviously in $L^2(\mathbb{R}^N)$. To show u is a bound state, we have to prove that $u\in L^2(\mathbb{R}^N)$. For this purpose, we need an integration estimate as [1, Lemma 17]. However, in our situation it seems hard to establish the desired estimate by choosing $R_n=n^{2/(2-\alpha)}$ as it was done in [1, Lemma 17]. In this paper, by technically taking $R_n=b^n$ for some $b>0$, we obtain an estimate in Lemma 3.1, which makes it possible to get a bound state for problem (1.1). Finally, as we have mentioned, in the asymptotically linear case we do not know whether the mountain pass solution is a ground state. So, we seek a ground state by looking for the minimizer of I over a related manifold.

The main results of this paper are as follows:

Theorem 1.1. *Assume that* (V_1), (F_1), (F_2), (K_1) *hold and* $\alpha\in(0,2)$ *in* (V_1). *Let* $l>\mu^*$ *with*

$$\mu^*=\inf\left\{\int_{\mathbb{R}^N}\left[|\nabla u|^2+V(x)u^2\right]dx\colon\ u\in H,\ \int_{\mathbb{R}^N}K(x)u^2\,dx=1\right\}.\tag{1.8}$$

Then problem (1.1) *has a solution* $u\in H$ *and it is a bound state, i.e.* $u\in H^1(\mathbb{R}^N)$.

If the functions V and K satisfy (V_1) and (K_2) with $0<\alpha<\beta$, it follows from Remark 1.1 and (1.2) that $\lim_{|x|\to+\infty}V(x)/K(x)=+\infty$ and (K_1) holds. Using (K_2), we see that $\mu^*\in(0,+\infty)$ and it can be achieved. Then, it is not difficult to find a function f satisfying (F_1), (F_2) such that $l>\mu^*$. Hence, V, K and f satisfy all the assumptions of Theorem 1.1. We mention that to prove, in this situation, that a bounded (PS) sequence of the functional I converges to a nontrivial solution, we can simply use the compactness of the embedding $H\hookrightarrow L_K^{p+1}$, $\sigma<p<\frac{N+2}{N-2}$, i.e. [1, Theorem 5], and our compactness result of Lemma 2.7 seems not necessary. However, if we have only the condition (K_1), Lemma 2.7 is necessary since in this case (K_2) may not be true, see our Examples 1.1 and 1.2.

Now, we give another example such that all the assumptions of Theorem 1.1 are satisfied, but

$$\lim_{|x|\to+\infty}V(x)/K(x)<+\infty,$$

and (K_2) fails.

Example 1.3. Let $V(x)$, $K(x)$ and f be given by Example 1.2, we know that (V_1), (F_1), (F_2) and (K_1) are satisfied for any $R_0 > 0$. Moreover, $l = R_0$ in (F_2). To verify $l > \mu^*$, we have to choose a special $R_0 > 0$. Indeed, for $R > 0$, taking $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\phi(x) = 1$ if $|x| \leqslant R$, $\phi(x) = 0$ if $|x| \geqslant 2R$ and $|\nabla\phi(x)| \leqslant C/R$ for all $x \in \mathbb{R}^N$, where $C > 0$ is a constant independent of x, then by $\operatorname{supp}\phi \subset B_{2R}$ we have, for $R_0 > 2R$,

$$\frac{\int_{\mathbb{R}^N} V(x)\phi^2\,dx}{\int_{\mathbb{R}^N} K(x)\phi^2\,dx} \leqslant \frac{\int_{\mathbb{R}^N} V(x)\phi^2\,dx}{\int_{\mathbb{R}^N} \frac{1}{1+2R} V(x)\phi^2\,dx} = 1 + 2R \quad \text{and}$$

$$\frac{\int_{\mathbb{R}^N} |\nabla\phi|^2\,dx}{\int_{\mathbb{R}^N} K(x)\phi^2\,dx} \leqslant \frac{\frac{C^2}{R^2}|B_{2R}|}{\int_{B_R} K(x)\,dx} \leqslant \frac{\frac{C^2}{R^2}|B_{2R}|}{\frac{1}{(1+R)\ln(3+R)}|B_R|} = \frac{C_1(1+R)\ln(3+R)}{R^2},$$

where $C_1 > 0$ is a constant independent of R. Let $R > 0$ large enough such that $C_1(1+R)\ln(3+R)/R^2 \leqslant 1$. Then for μ^* defined in (1.8), we have $\mu^* \leqslant 2R + 2$. Taking $R_0 = 2R + 3$, we see that

$$\lim_{s\to+\infty} \frac{f(s)}{s} = l = R_0 > \mu^*.$$

Remark 1.2. (i) If $K(x) \equiv 1$ and (V_2), (V_3) hold, then (K_1) is equivalent to

$$\sup\left\{\frac{f(s)}{s}\colon s > 0\right\} < V(\infty) := \liminf_{|x|\to+\infty} V(x).$$

In this situation, our result is similar to that of [13], or [20].

(ii) If $V(x) \equiv \lambda > 0$ and $\lim_{|x|\to+\infty} K(x) = K(\infty) = 0$, Theorem 1.1 becomes Theorem 2.4 of [8], but $K(\infty)$ cannot be zero in [8] and the monotonicity of $f(s)/s$ is also required there.

As pointed out in [1], if we want to find a ground state for problem (1.1) with $f(u) = u^p$, then $\sigma < p < \frac{N+2}{N-2}$ is a necessary condition. However, the following theorem shows that if f is not superlinear at infinity, problem (1.1) may have a ground state.

Theorem 1.2. *Under the assumptions of Theorem* 1.1*, problem* (1.1) *has a ground state* $\tilde{u} \in H^1(\mathbb{R}^N)$.

Notation. Throughout this paper, $|u|_p$ denotes the standard norm of $L^p(\mathbb{R}^N)$ for $1 \leqslant p \leqslant \infty$. $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{R}^N$. $B_R = \{x \in \mathbb{R}^N\colon |x| < R\}$, $B_R(y) = \{x \in \mathbb{R}^N\colon |x - y| < R\}$ and $kB_R(y) = \{kx\colon x \in B_R(y)\}$ for any constant $k > 0$.

2. Critical point in the weighted Sobolev space

In this section, our aim is to get a nonzero critical point of the functional I defined by (1.4). For this purpose, we use a variant version of the mountain pass theorem, which allows us to find a so-called Cerami type (PS) sequence. The properties of this kind of (PS) sequence are very helpful in showing the boundedness of the sequence in the asymptotically linear case. Let us recall this theorem, its proof can be found in Chapter IV of [10].

Proposition 2.1 *(Mountain Pass Theorem). Let E be a real Banach space with its dual space E^* and suppose that $I \in C^1(E, \mathbb{R})$ satisfies*

$$\max\{I(0), I(e)\} \leqslant \mu < \eta \leqslant \inf_{\|u\|=\rho} I(u),$$

for some $\mu < \eta$, $\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \geqslant \eta$ be characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leqslant \tau \leqslant 1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1], E)\colon\ \gamma(0) = 0,\ \gamma(1) = e\}$ is the set of continuous paths joining 0 and e, then there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \xrightarrow{n} c \geqslant \eta \quad \text{and} \quad (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \xrightarrow{n} 0.$$

This kind of sequence is usually called a *Cerami sequence*.

In what follows, we give first Lemmas 2.1 and 2.2 which ensure that the functional I has what is called the mountain pass geometry.

Lemma 2.1. *If the conditions* (V_1), (F_1), (F_2), (K_1) *hold, then there exist $\rho > 0$, $\eta > 0$ such that* $\inf\{I(u)\colon\ u \in H \text{ with } \|u\|_H = \rho\} > \eta$.

Proof. For any $\varepsilon > 0$, it follows from (F_1), (F_2) that there exists $C_\varepsilon > 0$ such that

$$|f(s)| \leqslant \varepsilon|s| + C_\varepsilon|s|^{2^*-1}, \quad \text{for all } s \in \mathbb{R},$$

where $2^* := \frac{2N}{N-2}$, and then,

$$|F(s)| \leqslant \frac{\varepsilon}{2}|s|^2 + \frac{C_\varepsilon}{2^*}|s|^{2^*}, \quad \text{for all } s \in \mathbb{R}. \tag{2.1}$$

From (1.5), (2.1) and the Sobolev inequality, we have for any $u \in H$,

$$\begin{aligned} \left|\int_{\mathbb{R}^N} K(x)F(u)\,dx\right| &\leqslant \frac{C_1\varepsilon}{2}\int_{\mathbb{R}^N} V(x)u^2\,dx + \frac{AC_1C_\varepsilon}{2^*}\int_{\mathbb{R}^N} |u|^{2^*}\,dx \\ &\leqslant \frac{C_1\varepsilon}{2}\|u\|_H^2 + \tilde{C}_\varepsilon\|u\|_H^{2^*}. \end{aligned} \tag{2.2}$$

This yields

$$I(u) \geqslant \frac{1 - C_1\varepsilon}{2}\|u\|_H^2 - \tilde{C}_\varepsilon\|u\|_H^{2^*}.$$

So, by fixing $\varepsilon \in (0, C_1^{-1})$ and letting $\|u\|_H = \rho > 0$ small enough, it is easy to see that there is $\eta > 0$ such that this lemma holds. □

Lemma 2.2. *Let* (V_1), (F_1), (F_2), (K_1) *hold and* $l > \mu^*$. *Then there exists* $v \in H$ *with* $\|v\|_H > \rho$ *such that* $I(v) < 0$, *where* ρ *is given by Lemma* 2.1.

Proof. By the definition of μ^* and $l > \mu^*$, there is $\phi \in H$ such that $\phi \geqslant 0$, $\int_{\mathbb{R}^N} K(x)\phi^2\,dx = 1$ and $\mu^* \leqslant \|\phi\|_H^2 < l$.

Then, by (F_2) and Fatou's lemma we deduce that

$$\lim_{t\to\infty} \frac{I(t\phi)}{t^2} = \frac{1}{2}\|\phi\|_H^2 - \lim_{t\to\infty} \int_{\mathbb{R}^N} K(x)\frac{F(t\phi)}{t^2}\,dx \leqslant \frac{1}{2}\big(\|\phi\|_H^2 - l\big) < 0,$$

and the lemma is proved by taking $v = t_0\phi$ with $t_0 > 0$ large. □

Based on Lemmas 2.1 and 2.2, Proposition 2.1 implies that there is a sequence $\{u_n\} \subset H$ such that

$$I(u_n) \xrightarrow{n} c > 0 \quad \text{and} \quad \big\|I'(u_n)\big\|_{H^{-1}}\big(1 + \|u_n\|_H\big) \xrightarrow{n} 0, \tag{2.3}$$

where H^{-1} denotes the dual space of H.

Next, we establish some preliminary results, i.e. Lemmas 2.3 to 2.5, which are used to prove that the above sequence $\{u_n\}$ is bounded in H.

Lemma 2.3. *Suppose that* $V(x)$ *satisfies* (V_1) *and* $0 < \alpha \leqslant 2$. *Then* $C_0^\infty(\mathbb{R}^N)$ *is dense in* $(H, \|\,.\,\|_H)$.

Proof. Motivated by the proof of Theorem 7.22 in [15], we show first that $C_0^\infty(\mathbb{R}^N)$ is dense in $H_0 := \{f \in H\colon\ f$ has a compact support$\}$, where we mean that a function f has a compact support if there is $R_f > 0$ such that $f(x) = 0$ a.e. for $|x| \geqslant R_f$. Then, for any $f \in H_0$, we have $f \in H^1(\mathbb{R}^N)$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, then for any $\epsilon > 0$, there is $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\|f - \varphi\|_{H^1} \leqslant \epsilon,$$

where $\|\,.\,\|_{H^1}$ denotes the standard norm of $H^1(\mathbb{R}^N)$. By (V_1), there exists $C > 0$ such that

$$\|f - \varphi\|_H \leqslant C\|f - \varphi\|_{H^1} \leqslant C\epsilon.$$

Thus, $C_0^\infty(\mathbb{R}^N)$ is dense in $(H_0, \|\,.\,\|_H)$.

Next, we claim that H_0 is dense in H. Choose $g \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leqslant g(x) \leqslant 1$ for all $x \in \mathbb{R}^N$, $g(x) \equiv 1$ for $|x| \leqslant 1$, and $g \equiv 0$ for $|x| \geqslant 2$. Let $g_m(x) = g(\frac{x}{m})$ for $m \in \mathbb{N}$. Then it is easy to see that for any $f \in H$, we have

$$\int_{\mathbb{R}^N} \big(g_m(x) - 1\big)^2 f^2(x)V(x)\,dx \xrightarrow{m} 0.$$

Since

$$\nabla(g_m f) = f\nabla g_m + g_m\nabla f,$$

we have

$$\left\|\nabla(g_m f)-\nabla f\right\|_{L^2(\mathbb{R}^N)}\leqslant\left\|(g_m-1)\nabla f\right\|_{L^2(\mathbb{R}^N)}+\|f\nabla g_m\|_{L^2(\mathbb{R}^N)}.$$

Clearly,

$$\left\|(g_m-1)\nabla f\right\|_{L^2(\mathbb{R}^N)}\xrightarrow{m}0.$$

From the definition of $g_m(x)$, we have

$$\left|f(x)\nabla g_m(x)\right|^2\leqslant C_2\left|f(x)\right|^2V(x),\quad \text{for all } x\in\mathbb{R}^N,$$

where $C_2=\sup_{m\geqslant 1}\frac{1+(2m)^\alpha}{m^2}|\nabla g(x)|_\infty^2$. Note that $0<\alpha\leqslant 2$, then $C_2<+\infty$. Hence, by the dominated convergence theorem we get

$$f\nabla g_m\xrightarrow{m}0\quad \text{strongly in } L^2\left(\mathbb{R}^N\right).$$

So,

$$g_m f\xrightarrow{m}f\quad \text{strongly in } H,$$

and H_0 is dense in H. □

Let $\{u_n\}$ be given by (2.3), and define $w_n:=u_n\|u_n\|_H^{-1}$. Clearly, w_n is bounded in H and there is $w\in H$ such that, up to a subsequence,

$$\begin{gathered}w_n\xrightarrow{n}w\quad\text{weakly in } H,\qquad w_n\xrightarrow{n}w\quad\text{a.e. in }\mathbb{R}^N,\\ w_n\xrightarrow{n}w\quad\text{strongly in } L^2_{\mathrm{loc}}\left(\mathbb{R}^N\right).\end{gathered}\tag{2.4}$$

For the above w we have the following lemma.

Lemma 2.4. *Assume that* (V_1), (F_1), (F_2), (K_1) *hold and* $\alpha\in(0,2]$, $l>\mu^*$. *If* $\|u_n\|_H\xrightarrow{n}+\infty$, *then* w *given by* (2.4) *is a nontrivial non-negative solution of*

$$-\Delta u(x)+V(x)u(x)=lK(x)u,\quad u\in H.\tag{2.5}$$

Proof. We prove this lemma through the following three steps.

Step 1. $w\not\equiv 0$.

By contradiction, if $w\equiv 0$, we claim that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}K(x)\frac{f(u_n)}{u_n}w_n^2\,dx<1.\tag{2.6}$$

If (2.6) is true, then it leads to a contradiction immediately. Indeed, since $\|u_n\|_H \xrightarrow{n} \infty$, it follows from (2.3) that

$$\langle I'(u_n), u_n\rangle / \|u_n\|_H^2 = o(1),$$

that is,

$$o(1) = \|w_n\|_H^2 - \int\limits_{R^N} K(x)\frac{f(u_n)}{u_n} w_n^2\,dx = 1 - \int\limits_{R^N} K(x)\frac{f(u_n)}{u_n} w_n^2\,dx,$$

where, and in what follows, $o(1)$ denotes a quantity which goes to zero as $n \to +\infty$. Clearly, this contradicts (2.6). Hence $w \not\equiv 0$ and step 1 is proved.

Now, we turn to showing that (2.6) holds. By (K_1), there is a constant $\eta \in (0, 1)$ such that

$$\sup\left\{\frac{f(s)}{s}\colon\ s > 0\right\} < \eta \inf\left\{\frac{V(x)}{K(x)}\colon\ |x| \geqslant R_0\right\}. \tag{2.7}$$

This yields, for all $n \in \mathbb{N}$,

$$\int\limits_{|x|\geqslant R_0} K(x)\frac{f(u_n)}{u_n}|w_n|^2\,dx \leqslant \eta \int\limits_{|x|\geqslant R_0} V(x)|w_n|^2\,dx \leqslant \eta < 1. \tag{2.8}$$

On the other hand, since the embedding $H^1(B_{R_0}) \hookrightarrow L^2(B_{R_0})$ is compact, $w_n \xrightarrow{n} w$ strongly in $L^2(B_{R_0})$. Passing to a subsequence, there exists $h \in L^2(B_{R_0})$ such that, for all $n \in \mathbb{N}$,

$$\left|w_n(x)\right| \leqslant h(x) \quad \text{a.e. in } B_{R_0}.$$

By (F_1), (F_2), there exists $C_3 > 0$ such that

$$\frac{f(t)}{t} \leqslant C_3, \quad \text{for all } t \in \mathbb{R}. \tag{2.9}$$

Then, for all $n \in \mathbb{N}$,

$$0 \leqslant K(x)\frac{f(u_n)}{u_n} w_n^2 \leqslant C_3 K(x) w_n^2(x) \leqslant C_3 |K|_\infty h^2(x) \quad \text{a.e. in } B_{R_0}. \tag{2.10}$$

Noting that $w_n \xrightarrow{n} w \equiv 0$ a.e. in $\mathbb{R}^N$, we get

$$K(x)\frac{f(u_n)}{u_n} w_n^2 \xrightarrow{n} 0 \quad \text{a.e. in } B_{R_0}. \tag{2.11}$$

It follows from (2.10), (2.11) and the dominated convergence theorem that

$$\lim_{n\to\infty} \int\limits_{|x|<R_0} K(x)\frac{f(u_n)}{u_n} w_n^2\,dx = 0. \tag{2.12}$$

Hence, (2.6) is deduced from (2.8) and (2.12).

Step 2. $w \geqslant 0$.

Let $w_n^-(x) = \max\{-w_n(x), 0\}$, w_n^- is also bounded in H. If $\|u_n\|_H \xrightarrow{n} \infty$, then

$$\frac{\langle I'(u_n), w_n^- \rangle}{\|u_n\|_H} = o(1),$$

that is,

$$-\|w_n^-\|_H^2 = \int_{\mathbb{R}^N} K(x) \frac{f(u_n)}{\|u_n\|_H} w_n^- \, dx + o(1). \tag{2.13}$$

By (F$_1$), $f(t) \equiv 0$ for all $t \leqslant 0$. It follows from (2.13) that $\|w_n^-\|_H = o(1)$. Thus $w^- = 0$ a.e. in $x \in \mathbb{R}^N$ and $w \geqslant 0$.

Step 3. w solves (2.5).

By Lemma 2.3, it is sufficient to prove that for any $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \big[\nabla w(x) \nabla \phi(x) + V(x) w(x) \phi(x)\big] \, dx = \int_{\mathbb{R}^N} l K(x) w(x) \phi(x) \, dx. \tag{2.14}$$

Using (2.3) and $\|u_n\|_H \xrightarrow{n} \infty$, we have

$$\frac{\langle I'(u_n), \phi \rangle}{\|u_n\|_H} = o(1), \quad \text{for any } \phi \in C_0^\infty\big(\mathbb{R}^N\big),$$

that is,

$$\int_{\mathbb{R}^N} \big[\nabla w_n \nabla \phi + V(x) w_n \phi\big] \, dx = \int_{\mathbb{R}^N} K(x) \frac{f(u_n)}{u_n} w_n \phi \, dx + o(1). \tag{2.15}$$

Since $w_n \overset{n}{\rightharpoonup} w$ weakly in H, we see that

$$\int_{\mathbb{R}^N} \big[\nabla w \nabla \phi + V(x) w \phi\big] \, dx = \int_{\mathbb{R}^N} K(x) \frac{f(u_n)}{u_n} w_n \phi \, dx + o(1). \tag{2.16}$$

So, step 3 is complete provided that

$$\int_{\mathbb{R}^N} K(x) \frac{f(u_n)}{u_n} w_n(x) \phi(x) \, dx \xrightarrow{n} \int_{\mathbb{R}^N} l K(x) w(x) \phi(x) \, dx. \tag{2.17}$$

In fact, by (2.9) and (1.5) we have

$$\int_{\mathbb{R}^N} \left| K^{\frac{1}{2}}(x) \frac{f(u_n)}{u_n} w_n(x) \right|^2 dx \leqslant C \int_{\mathbb{R}^N} V(x) w_n^2 \, dx \leqslant C \|w_n\|_H^2 \leqslant C, \tag{2.18}$$

that is, $\{K^{\frac{1}{2}}(x) \frac{f(u_n)}{u_n} w_n(x)\}$ is bounded in $L^2(\mathbb{R}^N)$.

Let

$$\Omega_+ = \{x \in \mathbb{R}^N\colon\ w(x) > 0\} \quad \text{and} \quad \Omega_0 = \{x \in \mathbb{R}^N\colon\ w(x) = 0\}.$$

Noting that

$$w_n(x) = \frac{u_n(x)}{\|u_n\|_H} \xrightarrow{n} w(x) \quad \text{a.e. } x \in \mathbb{R}^N \quad \text{and} \quad \|u_n\|_H \xrightarrow{n} +\infty,$$

then $u_n(x) \xrightarrow{n} +\infty$ a.e. in $x \in \Omega_+$. Hence by (F_2), we have

$$K^{\frac{1}{2}}(x)\frac{f(u_n)}{u_n}w_n(x) \xrightarrow{n} lK^{\frac{1}{2}}(x)w(x) \quad \text{a.e. in } x \in \Omega_+.$$

Since $w_n(x) \xrightarrow{n} 0$ a.e. in $x \in \Omega_0$, it follows from (2.9) that

$$K^{\frac{1}{2}}(x)\frac{f(u_n)}{u_n}w_n(x) \xrightarrow{n} 0 \equiv lK^{\frac{1}{2}}(x)w(x) \quad \text{a.e. in } x \in \Omega_0.$$

These and (2.18) imply that

$$K^{\frac{1}{2}}(x)\frac{f(u_n)}{u_n}w_n(x) \xrightharpoonup{n} lK^{\frac{1}{2}}(x)w(x) \quad \text{weakly in } L^2(\mathbb{R}^N). \tag{2.19}$$

From $\phi \in C_0^\infty(\mathbb{R}^N)$ and $K \in L^\infty(\mathbb{R}^N)$, we know that $K^{\frac{1}{2}}(x)\phi \in L^2(\mathbb{R}^N)$, then (2.19) leads to (2.17). □

Lemma 2.5. *If* $0 < \alpha \leqslant 2$, $l > \mu^*$ *and* (V_1), (K_1) *hold, then problem* (2.5) *has no nontrivial non-negative solutions.*

Proof. Since $l > \mu^*$, there is a constant $\delta > 0$ such that $\mu^* < \mu^* + \delta < l$. By the definition of μ^* in (1.8), there exists $v_\delta \in H$ such that $\int_{R^N} K(x)v_\delta^2(x)\,dx = 1$ and

$$\mu^* \leqslant \|v_\delta\|_H^2 < \mu^* + \delta.$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in H by Lemma 2.3, we may assume $v_\delta \in C_0^\infty(\mathbb{R}^N)$. Motivated by [13], let $R > 0$ be such that $\operatorname{supp} v_\delta \subset B_R$ and define

$$\mu_R = \inf\left\{ \int_{B_R} \left[|\nabla u|^2(x) + V(x)u^2(x)\right]dx\colon \int_{B_R} K(x)u^2(x)\,dx = 1,\ u \in H_0^1(B_R)\right\}.$$

Then, $v_\delta \in H_0^1(B_R)$ and

$$\mu_R \leqslant \|v_\delta\|_H^2 < \mu^* + \delta < l. \tag{2.20}$$

By the compactness of the embedding $H_0^1(B_R) \hookrightarrow L^2(B_R)$, it is not difficult to see that there exists $w_R \in H_0^1(B_R) \setminus \{0\}$ with $w_R \geqslant 0$ and $\int_{B_R} K(x) w_R^2(x)\,dx = 1$ such that

$$-\Delta w_R + V(x) w_R = \mu_R K(x) w_R, \quad x \in B_R. \tag{2.21}$$

It follows from the strong maximum principle that

$$w_R(x) > 0, \quad \forall x \in B_R, \qquad \frac{\partial w_R(x)}{\partial \nu} < 0, \quad \forall |x| = R.$$

Therefore, if $0 \not\equiv u \in H$ is a non-negative solution of (2.5), then

$$\begin{aligned} \mu_R \int_{B_R} K(x) w_R u\,dx &= \int_{B_R} \big(-\Delta w_R + V(x) w_R\big) u\,dx \\ &= \int_{B_R} \nabla u \nabla w_R + \int_{B_R} V(x) u w_R\,dx - \int_{\partial B_R} \frac{\partial w_R}{\partial \nu} u\,d\sigma \\ &= \int_{B_R} l K(x) u w_R\,dx - \int_{\partial B_R} \frac{\partial w_R}{\partial \nu} u\,d\sigma \\ &\geqslant l \int_{B_R} K(x) u w_R\,dx. \end{aligned} \tag{2.22}$$

Using $u \geqslant 0$ and $u \not\equiv 0$, we may choose $R > 0$ large enough such that $\int_{B_R} K(x) u w_R\,dx > 0$. So, (2.22) implies that $\mu_R \geqslant l$. This contradicts (2.20). □

Now, we can prove that the Cerami sequence $\{u_n\}$ in (2.3) is bounded in H. In fact, if $\{u_n\}$ is not bounded in H, we may assume that $\|u_n\|_H \xrightarrow{n} +\infty$. Define w_n as in (2.4), then it is easy to get a contradiction by simply using Lemmas 2.4 and 2.5. So we have

Lemma 2.6. *Under conditions* (V_1), (F_1), (F_2), (K_1) *and* $0 < \alpha \leqslant 2$, *then the sequence* $\{u_n\}$ *given in* (2.3) *is bounded in* H *if* $l > \mu^*$.

To prove that the Cerami sequence $\{u_n\}$ in (2.3) converges to a nonzero critical point of I, the following compactness lemma is useful.

Lemma 2.7. *Let* (V_1), (F_1), (F_2), (K_1) *hold and* $0 < \alpha < 2$. *Then for any* $\epsilon > 0$, *there exist* $R(\epsilon) > R_0$ *and* $n(\epsilon) > 0$ *such that*

$$\int_{|x| \geqslant R} \big[|\nabla u_n|^2 + V(x) u_n^2\big]\,dx \leqslant \epsilon, \tag{2.23}$$

for all $R \geqslant R(\epsilon)$ *and* $n \geqslant n(\epsilon)$, *where* R_0 *is given by* (K_1) *and the sequence* $\{u_n\}$ *is given by* (2.3).

Proof. For R_0 given by (K_1), define

$$C_1(R_0,\alpha,a) := \sup\left\{\frac{1+(2R)^\alpha}{aR^2}\colon\ R \geqslant R_0\right\} = \frac{1+(2R_0)^\alpha}{aR_0^2} \quad \text{and} \tag{2.24}$$

$$C_2(R_0,\alpha,a) := \sup\left\{\frac{1+(2R)^\alpha}{aR^\alpha}\colon\ R \geqslant R_0\right\} = \frac{1+(2R_0)^\alpha}{aR_0^\alpha}, \tag{2.25}$$

where α and a are given by (V_1). Then, by (V_1), (2.24) and (2.25), we have, for all $R \geqslant R_0$,

$$1/R^2 \leqslant C_1(R_0,\alpha,a)V(x), \quad \text{for all } |x| \leqslant 2R, \tag{2.26}$$

and

$$1/R^\alpha \leqslant C_2(R_0,\alpha,a)V(x), \quad \text{for all } |x| \leqslant 2R. \tag{2.27}$$

Let $\xi_R : \mathbb{R}^N \to [0,1]$ be a smooth function such that

$$\xi_R(x) = \begin{cases} 0, & 0 \leqslant |x| \leqslant R, \\ 1, & |x| \geqslant 2R, \end{cases} \tag{2.28}$$

and, for some constant $C_0 > 0$ (independent of R),

$$\left|\nabla \xi_R(x)\right| \leqslant \frac{C_0}{R}, \quad \text{for all } x \in \mathbb{R}^N. \tag{2.29}$$

Then, by (2.26), for all $n \in \mathbb{N}$ and $R \geqslant R_0$, we have

$$\begin{aligned} \int\limits_{\mathbb{R}^N} \left|\nabla(u_n\xi_R)\right|^2 dx &\leqslant 2\int\limits_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \frac{2C_0^2}{R^2}\int\limits_{R\leqslant|x|\leqslant 2R} |u_n|^2\,dx \\ &\leqslant \left[2 + 2C_0^2 C_1(R_0,\alpha,a)\right]\|u_n\|_H^2. \end{aligned} \tag{2.30}$$

This implies that

$$\|u_n\xi_R\|_H \leqslant \left[3 + 2C_0^2C_1(R_0,\alpha,a)\right]^{\frac{1}{2}}\|u_n\|_H, \tag{2.31}$$

for all $n \in \mathbb{N}$ and $R \geqslant R_0$. By $0 < \alpha < 2$, for any $\epsilon > 0$, there exists $R(\epsilon) \geqslant R_0$ such that

$$R^{\alpha-2} \leqslant \frac{4\epsilon^2}{C_0^2C_2(R_0,\alpha,a)}, \quad \text{for all } R \geqslant R(\epsilon). \tag{2.32}$$

By (2.3), $\|I'(u_n)\|_{H^{-1}}\|u_n\|_H \xrightarrow{n} 0$, so for any $\epsilon > 0$, there exists $n(\epsilon) > 0$ such that

$$\|u_n\|_H\left\|I'(u_n)\right\|_{H^{-1}} \leqslant \frac{\epsilon}{[3 + 2C_0^2C_1(R_0,\alpha,a)]^{\frac{1}{2}}}, \quad \text{for all } n \geqslant n(\epsilon). \tag{2.33}$$

Hence, it follows from (2.31) and (2.33) that

$$\left|\left\langle I'(u_n), u_n\xi_R\right\rangle\right| \leqslant \left\| I'(u_n)\right\|_{H^{-1}} \|u_n\xi_R\|_H \leqslant \epsilon, \tag{2.34}$$

for all $n \geqslant n(\varepsilon)$ and $R \geqslant R_0$. Note that

$$\begin{aligned}\left\langle I'(u_n), u_n\xi_R\right\rangle = & \int\limits_{\mathbb{R}^N} |\nabla u_n|^2 \xi_R\,dx + \int\limits_{\mathbb{R}^N} V(x)u_n^2\xi_R\,dx \\ & + \int\limits_{\mathbb{R}^N} u_n\nabla u_n\nabla\xi_R\,dx - \int\limits_{\mathbb{R}^N} K(x)f(u_n)u_n\xi_R\,dx.\end{aligned} \tag{2.35}$$

For $R \geqslant R(\epsilon)$, using (2.27) and (2.32), we have,

$$\frac{C_0^2 C_2(R_0,\alpha,a)}{R^2} \leqslant 4\epsilon^2 \frac{1}{R^\alpha} \leqslant 4\epsilon^2 C_2(R_0,\alpha,a)V(x), \quad \text{for all } |x| \leqslant 2R,$$

that is,

$$\frac{C_0^2}{R^2} \leqslant 4\epsilon^2 V(x), \quad \text{for all } |x| \leqslant 2R. \tag{2.36}$$

Therefore, from (2.29) and (2.36), we get, for all $n \in \mathbb{N}$ and $R \geqslant R(\epsilon)$,

$$\begin{aligned}\int\limits_{\mathbb{R}^N} |u_n\nabla u_n\nabla\xi_R|\,dx &\leqslant \epsilon \int\limits_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \frac{1}{4\epsilon}\int\limits_{|x|\leqslant 2R} u_n^2\frac{C_0^2}{R^2}\,dx \\ &\leqslant \epsilon \int\limits_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \epsilon \int\limits_{|x|\leqslant 2R} V(x)u_n^2\,dx \\ &\leqslant \epsilon\|u_n\|_H^2.\end{aligned} \tag{2.37}$$

By (F_1), (K_1) and (2.28), there exists $\eta \in (0,1)$ such that, for all $n \in \mathbb{N}$ and $R \geqslant R_0$

$$\int\limits_{\mathbb{R}^N} \left|K(x)f(u_n)u_n\xi_R\right|dx \leqslant \eta \int\limits_{\mathbb{R}^N} V(x)u_n^2\xi_R\,dx. \tag{2.38}$$

Combining (2.35), (2.37) and (2.38), for all $n \in \mathbb{N}$ and $R \geqslant R(\epsilon) \geqslant R_0$, we see that

$$\left|\left\langle I'(u_n), u_n\xi_R\right\rangle\right| \geqslant \int\limits_{\mathbb{R}^N} |\nabla u_n|^2\xi_R\,dx + (1-\eta)\int\limits_{\mathbb{R}^N} V(x)u_n^2\xi_R\,dx - \epsilon\|u_n\|_H^2. \tag{2.39}$$

Since $\{\|u_n\|_H\}$ is bounded by Lemma 2.6, it follows from (2.34) and (2.39) that there exists $C_3 > 0$ such that, for all $n \geqslant n(\epsilon)$ and $R \geqslant R(\epsilon)$,

$$\int\limits_{\mathbb{R}^N} |\nabla u_n|^2\xi_R\,dx + (1-\eta)\int\limits_{\mathbb{R}^N} V(x)u_n^2\xi_R\,dx \leqslant C_3\epsilon. \tag{2.40}$$

from $\eta \in (0,1)$ and (2.28), it is easy to see that (2.40) implies (2.23). □

Finally, we prove the main theorem of this section.

Theorem 2.1. *Let* $0 < \alpha < 2$*,* $l > \mu^*$ *and* (V_1)*,* (F_1)*,* (F_2)*,* (K_1) *hold. Then* I *has a nonzero critical point in* H*.*

Proof. By Lemma 2.6, the sequence $\{u_n\}$ in (2.3) is bounded in H. We may assume that, up to a subsequence, $u_n \xrightharpoonup{n} u$ weakly in H for some $u \in H$. In order to prove our theorem, it is now sufficient to show that $\|u_n\|_H \xrightarrow{n} \|u\|_H$. By (2.3),

$$\langle I'(u_n), u_n\rangle = \int\limits_{\mathbb{R}^N} |\nabla u_n|^2 + V(x)|u_n|^2\,dx - \int\limits_{\mathbb{R}^N} Kf(u_n)u_n = o(1)$$

and

$$\langle I'(u_n), u\rangle = \int\limits_{\mathbb{R}^N} \nabla u_n \nabla u + V(x)u_n u\,dx - \int\limits_{\mathbb{R}^N} Kf(u_n)u = o(1),$$

so to show $\|u_n\|_H \xrightarrow{n} \|u\|_H$ is equivalent to proving that

$$\int\limits_{\mathbb{R}^N} K(x)f(u_n)(u_n - u)\,dx = o(1). \tag{2.41}$$

For any $\epsilon > 0$, by Lemma 2.7 and for n large enough, we have

$$\begin{aligned}
&\int\limits_{|x|\geqslant R(\epsilon)} K(x)f(u_n)(u_n-u)\,dx \\
&\quad\leqslant \Biggl(\int\limits_{|x|\geqslant R(\epsilon)} K(x)|u_n-u|^2\,dx\Biggr)^{\frac{1}{2}}\Biggl(\int\limits_{|x|\geqslant R(\epsilon)} K(x)\bigl|f(u_n)\bigr|^2\,dx\Biggr)^{\frac{1}{2}} \\
&\quad\leqslant C\Biggl(\int\limits_{|x|\geqslant R(\epsilon)} V(x)|u_n-u|^2\,dx\Biggr)^{\frac{1}{2}}\Biggl(\int\limits_{|x|\geqslant R(\epsilon)} V(x)|u_n|^2\,dx\Biggr)^{\frac{1}{2}} \\
&\quad\leqslant C\epsilon.
\end{aligned} \tag{2.42}$$

This and the compactness of the embedding $H \hookrightarrow L^2_{\mathrm{loc}}(\mathbb{R}^N)$ imply (2.41). □

3. Existence of a bound state and a ground state

The aim of this section is to prove Theorems 1.1 and 1.2. For Theorem 1.1, it is sufficient to show that the nonzero critical point, $u \in H$, of I obtained by Theorem 2.1 is a bound state, that is, $u \in L^2(\mathbb{R}^N)$. For this purpose, we need to establish some integration estimates on u, which are essentially motivated by [1]. However, we should mention that in the proof of the following Lemma 3.1, it is crucial to choose R_n properly. It seems very difficult to get the desired estimate if we use the same R_n as in [1, Lemma 17]. To prove Theorem 1.2, we use a constrained minimization technique.

Lemma 3.1. *Suppose that* (V_1), (F_1), (F_2), (K_1) *hold. Let* $0<\alpha<2$ *and* $u\in H$ *be the critical point obtained in Theorem* 2.1. *Then there exist* η_0, $\delta\in(0,1)$ *and* $n(\alpha)>0$ *such that, for all* $n\geqslant n(\alpha)$,

$$\int_{\Omega_{n+1}} |\nabla u|^2+V(x)u^2\,dx\leqslant\big[\eta_0+\delta(1-\eta_0)\big]\int_{\Omega_n}|\nabla u|^2+V(x)u^2\,dx, \tag{3.1}$$

where

$$\Omega_n=\mathbb{R}^N\setminus B_{R_n},\quad R_n=b^n,\quad b=\left(\frac{1}{\eta_0}\right)^{\frac{1}{3}}. \tag{3.2}$$

Proof. Since $0<\alpha<2$, there is $\delta\in(0,1)$ such that $\frac{\alpha+1}{3}<1-\delta$. Noting that

$$\lim_{\eta\to1^-}\frac{\ln[\eta+\delta(1-\eta)]}{\frac{1}{3}\ln\frac{1}{\eta}}=-3(1-\delta)<-\alpha-1, \tag{3.3}$$

then, by (K_1), we may choose $\eta_0\in(0,1)$ such that

$$\sup\left\{\frac{f(s)}{s}\colon s>0\right\}<\eta_0\inf\left\{\frac{V(x)}{K(x)}\colon |x|\geqslant R_0\right\} \tag{3.4}$$

and

$$\frac{\ln[\eta_0+\delta(1-\eta_0)]}{\frac{1}{3}\ln\frac{1}{\eta_0}}<-\alpha-1.$$

Hence, there exists $\theta\in(0,1)$ such that

$$C(\delta,\eta_0,\theta):=\theta\frac{\ln[\eta_0+\delta(1-\eta_0)]}{\frac{1}{3}\ln\frac{1}{\eta_0}}<-\alpha-1. \tag{3.5}$$

Let $\xi_n:\mathbb{R}^N\to[0,1]$ be a smooth function such that

$$\xi_n(x)=\begin{cases}0, & 0\leqslant|x|\leqslant R_n,\\ 1, & |x|\geqslant R_{n+1},\end{cases} \tag{3.6}$$

where R_n is given by (3.2). Moreover, there exists $A_0>0$, independent of n, such that $|\nabla\xi_n(x)|\leqslant\frac{A_0}{R_{n+1}-R_n}$, for all $x\in\mathbb{R}^N$. Noting that $0<\alpha<2$ and $b>1$ (b is given by (3.2)), we see that

$$\lim_{n\to\infty}\frac{1+b^{\alpha(n+1)}}{b^{2(n+1)}}\left(\frac{A_0b}{b-1}\right)^2=0.$$

So, there exists $n(\alpha) > 0$ such that, for all $n \geqslant n(\alpha)$,

$$\frac{1 + b^{\alpha(n+1)}}{b^{2(n+1)}} \left(\frac{A_0 b}{b-1}\right)^2 \leqslant \left[2\delta(1-\eta_0)\right]^2 a, \tag{3.7}$$

where a is given by (V_1). From (3.7) and (V_1), for $R_n \leqslant |x| \leqslant R_{n+1}$ and $n \geqslant n(\alpha)$, we have

$$\begin{aligned} \left|\nabla \xi_n(x)\right|^2 &\leqslant A_0^2 |R_{n+1} - R_n|^{-2} = b^{-2(n+1)} \left(\frac{A_0 b}{b-1}\right)^2 \\ &\leqslant \frac{a}{1 + (R_{n+1})^\alpha} \left[2\delta(1-\eta_0)\right]^2 \leqslant V(x)\left[2\delta(1-\eta_0)\right]^2. \end{aligned}$$

Then by the definition of $\xi_n(x)$, we get, for all $n \geqslant n(\alpha)$ and $x \in \mathbb{R}^N$,

$$\left|\nabla \xi_n(x)\right|^2 \leqslant V(x)\left[2\delta(1-\eta_0)\right]^2. \tag{3.8}$$

Since $\langle I'(u), u\xi_n \rangle = 0$, by Young inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \xi_n \left[|\nabla u|^2 + V(x)u^2\right] dx &= \int_{\mathbb{R}^N} K(x) f(u) u \xi_n \, dx - \int_{\mathbb{R}^N} u \nabla u \nabla \xi_n \, dx \\ &\leqslant \int_{\Omega_n} \left|K(x) f(u) u\right| dx + \int_{\Omega_n} \epsilon |\nabla u|^2 \, dx \\ &\quad + \int_{\Omega_n} \frac{1}{4\epsilon} |\nabla \xi_n|^2 u^2 \, dx. \end{aligned} \tag{3.9}$$

Taking $\epsilon = \delta(1-\eta_0)$, then by (3.4) (3.8) and (3.9), we deduce that, for all $n \geqslant n(\alpha)$,

$$\begin{aligned} \int_{\Omega_{n+1}} \left[|\nabla u|^2 + V(x)u^2\right] dx &= \int_{\Omega_{n+1}} \xi_n \left[|\nabla u|^2 + V(x)u^2\right] dx \\ &\leqslant \int_{\Omega_n} \xi_n \left[|\nabla u|^2 + V(x)u^2\right] dx \\ &\leqslant \int_{\Omega_n} \left|K(x) f(u) u\right| dx + \delta(1-\eta_0) \int_{\Omega_n} \left[|\nabla u|^2 + V(x)u^2\right] dx \\ &\leqslant \left[\eta_0 + \delta(1-\eta_0)\right] \int_{\Omega_n} \left[|\nabla u|^2 + V(x)u^2\right] dx. \end{aligned}$$

This implies that (3.1) holds. $\square$

Lemma 3.2. *Let $n(\alpha)$ be given in Lemma* 3.1. *Then there exists $R(\alpha) > n(\alpha)$ such that, for all $\rho > R(\alpha)$,*

$$\int\limits_{|x|>\rho} \left[|\nabla u|^2 + V(x)u^2\right] dx \leqslant \|u\|_H^2 \exp\left\{C(\delta, \eta_0, \theta) \ln \rho\right\}, \tag{3.10}$$

where $C(\delta, \eta_0, \theta)$ is given by (3.5)*.*

Proof. Since

$$\lim_{R\to+\infty}\left[\frac{\ln R - \ln n(\alpha)}{\ln b} - 2\right] \Big/ \left(\frac{\theta \ln R}{\ln b}\right) = \frac{1}{\theta} > 1,$$

there exists a constant $R(\alpha) > n(\alpha)$ such that

$$\ln R(\alpha) - \ln n(\alpha) > 2 \ln b$$

and

$$\frac{\ln \rho - \ln n(\alpha)}{\ln b} - 2 \geqslant \frac{\theta \ln \rho}{\ln b} > 0, \quad \text{for all } \rho > R(\alpha). \tag{3.11}$$

Given $\rho > R(\alpha)$, there exist two integers $\bar{n}$ and $\tilde{n}$ such that

$$R_{\bar{n}} \leqslant n(\alpha) < R_{\bar{n}+1}, \qquad R_{\tilde{n}-1} \leqslant \rho < R_{\tilde{n}}.$$

Thus

$$\tilde{n} - \bar{n} > \frac{\ln \rho - \ln n(\alpha)}{\ln b} > 2. \tag{3.12}$$

By Lemma 3.1, (3.11) and (3.12), we deduce that

$$\begin{aligned}
\int\limits_{|x|>\rho} \left[|\nabla u|^2 + V(x)u^2\right] dx &\leqslant \int\limits_{|x|>R_{\tilde{n}-1}} \left[|\nabla u|^2 + V(x)u^2\right] dx \\
&\leqslant \left[\eta_0 + \delta(1-\eta_0)\right]^{\tilde{n}-\bar{n}-2} \int\limits_{|x|>R_{\bar{n}+1}} \left[|\nabla u|^2 + V(x)u^2\right] dx \\
&\leqslant \|u\|_H^2 \exp\left\{\ln\left[\eta_0 + \delta(1-\eta_0)\right]\left(\frac{\ln \rho - \ln n(\alpha)}{\ln b} - 2\right)\right\} \\
&\leqslant \|u\|_H^2 \exp\left\{C(\delta, \eta_0, \theta) \ln \rho\right\},
\end{aligned}$$

that is, (3.10) holds. □

Proof of Theorem 1.1. Let $r \geqslant 2$, $|y| \geqslant 2r$, then

$$\sup\left\{\frac{1+|x|^\alpha}{a|y|^\alpha}\colon x \in B_r(y)\right\} \leqslant \frac{1+(r+|y|)^\alpha}{a|y|^\alpha} \leqslant \sup\left\{\frac{1+(\frac{3}{2}|y|)^\alpha}{a|y|^\alpha}\colon |y| \geqslant 4\right\}.$$

Denote

$$C_3(\alpha) = \sup\left\{\frac{1+(\frac{3}{2}|y|)^\alpha}{a|y|^\alpha}\colon |y| \geqslant 4\right\} = \frac{1}{4^\alpha a} + \frac{3^\alpha}{2^\alpha a}.$$

Then

$$\sup\left\{\frac{1+|x|^\alpha}{a}\colon x \in B_r(y)\right\} \leqslant C_3(\alpha)|y|^\alpha. \tag{3.13}$$

Since $B_r(y) \subset \{x \in \mathbb{R}^N\colon |x| \geqslant \frac{|y|}{2}\}$, by Lemma 3.2 and (3.13) we have, for all $|y| > 2R(\alpha)$,

$$\begin{aligned}
\int\limits_{B_r(y)} u^2\,dx &\leqslant \int\limits_{B_r(y)} \frac{1+|x|^\alpha}{a} V(x)u^2\,dx \leqslant C_3(\alpha)|y|^\alpha \int\limits_{B_r(y)} V(x)u^2\,dx \\
&\leqslant C_3(\alpha)\,|y|^\alpha \int\limits_{|x| \geqslant \frac{|y|}{2}} V(x)u^2\,dx \\
&\leqslant C_3(\alpha)2^{-C(\delta,\eta_0,\theta)}|y|^\alpha \|u\|_H^2 |y|^{C(\delta,\eta_0,\theta)} \\
&= C_4\|u\|_H^2 |y|^{\alpha+C(\delta,\eta_0,\theta)},
\end{aligned} \tag{3.14}$$

where $C_4 = C_3(\alpha)2^{-C(\delta,\eta_0,\theta)}$. Let $m \in \mathbb{N}^+$ with $|y_i| \geqslant 2$ $(i = 1, 2, \ldots, m)$ be such that

$$B_5 \setminus B_2 \subset \bigcup_{i=1}^m B_1(y_i).$$

Let K_0 denote a positive integer such that $2^{K_0} > R(\alpha)$. Since $|2^k y_i| \geqslant 2^{k+1}$ for $k \geqslant 1$, by (3.14) we deduce that

$$\begin{aligned}
\int\limits_{|x|\geqslant 2} u^2\,dx &\leqslant \sum_{k=0}^\infty \int\limits_{2^k(B_5\setminus B_2)} u^2\,dx \leqslant \sum_{i=1}^m \sum_{k=0}^\infty \int\limits_{B_{2^k}(2^k y_i)} u^2\,dx \\
&\leqslant \sum_{i=1}^m \sum_{k=0}^{K_0-1} \int\limits_{B_{2^k}(2^k y_i)} u^2\,dx + C_4\|u\|_H^2 \sum_{i=1}^m \sum_{k=K_0}^\infty \left|2^k y_i\right|^{\alpha+C(\delta,\eta_0,\theta)}.
\end{aligned}$$

By (3.5), $C(\delta, \eta_0, \theta) + \alpha < -1$, then the above inequality implies that $\int_{|x| \geqslant 2} u^2\,dx < +\infty$ and $u \in L^2(\mathbb{R}^N)$. Thus $u \in H^1(\mathbb{R}^N)$ is a bound state of problem (1.1). $\square$

Finally, we prove Theorem 1.2, that is, problem (1.1) has a ground state.

Proof of Theorem 1.2. Set

$$\mathcal{N} = \big\{u \in H \setminus \{0\}\colon\ I'(u) = 0\big\}.$$

By Theorem 1.1, $\mathcal{N} \neq \emptyset$. We claim that there exists $M > 0$ such that

$$I(u) \geqslant -M, \quad \text{for all } u \in \mathcal{N}.$$

Otherwise, there exists $\{u_n\} \subset \mathcal{N}$ such that

$$I(u_n) < -n, \quad \text{for any } n \in \mathbb{N}. \tag{3.15}$$

It follows from (2.2) that

$$I(u_n) \geqslant \frac{1}{4}\|u_n\|_H^2 - C\|u_n\|_H^{2^*}. \tag{3.16}$$

This and (3.15) imply that $\|u_n\|_H \xrightarrow{n} +\infty$. Let $w_n = u_n\|u_n\|_H^{-1}$, then there is $w \in H$ such that, up to a subsequence, (2.4) holds. Note that $I'(u_n) = 0$ by $u_n \in \mathcal{N}$, as in the proof of Lemma 2.4, we see that w should be a nontrivial non-negative solution of (2.5), which is impossible by Lemma 2.5. Then, I is bounded from below on $\mathcal{N}$. So, we may define

$$\tilde{c} = \inf\big\{I(u)\colon\ u \in \mathcal{N}\big\},$$

and $\tilde{c} \geqslant -M$. Let $\{\tilde{u}_n\} \subset \mathcal{N}$ be such that $I(\tilde{u}_n) \xrightarrow{n} \tilde{c}$. It follows from (1.5) (F_1), (F_2) and $\langle I'(\tilde{u}_n), \tilde{u}_n\rangle = 0$ that there exists $\tilde{\rho} > 0$ such that

$$\liminf_{n \to +\infty} \|\tilde{u}_n\|_H \geqslant \tilde{\rho} > 0. \tag{3.17}$$

Following almost the same procedures as the proofs of Lemma 2.6 and Theorem 2.1 in Section 2, we can show that $\{\tilde{u}_n\}$ is bounded in H and it has a strongly convergent subsequence. By (3.17), there exists $\tilde{u} \in H \setminus \{0\}$ such that $\tilde{u}_n \xrightarrow{n} \tilde{u}$ strongly in H. Thus $I(\tilde{u}) = \tilde{c}$ and $I'(\tilde{u}) = 0$. Similar to the proof of Theorem 1.1, we may prove that $\tilde{u} \in L^2(\mathbb{R}^N)$. Therefore, $\tilde{u} \in H^1(\mathbb{R}^N)$ is a ground state of problem (1.1). $\square$

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